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Of Steam through Nozzles

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THE THEORY OF THE FLOW OF STEAM THROUGH NOZZLES

BY

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THIS IS TO CERTIFY THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

KENNETH GARDNER SMITH

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IS APPROVED BY ME AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE DEGREE

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THE THEORY OF THE

FLOW OF STEAM THROUGH NOZZLES.

Prefatory Note.

From a practical standpoint the following thesis may seem disappointing owing to the fact that very few experimental results are adduced in support of the theory. The results of actual experiments along this line will be found in the theses of Messrs. C. F. Dosch and F. W. Marquis of the class of 1905, who have investigated the impulse and velocity of the steam jet respectively. The two factors of impulse, mass and velocity, have been constantly kept in mind and it is with the variation of these two factors as affected by the form of the nozzle that the following thesis deals. Neither the theory nor experiments are in anyway complete or conclusive and all that the writers have hoped to do is to point out some of the problems as yet unsolved in this new and interesting field.

CHAPTER 1.

Historical Outline of the Development of the Laws of the Flow of Elastic Fluids.

Since the advent of the steam turbine the phenomena of steam flow, steam velocity, and steam impact have received considerable attention both from the theoretical and experimental side. Although the theory is, as yet, far from complete and satisfactory and the experimental work fragmentary, it is the aim of the following thesis, so far as possible, to unite into a more or less connected whole the laws of flow, so far as known, and the results of the most recent investigations along this line. The theory of steam flow depends primarily on the theory of the flow of elastic fluids and a short historical sketch will aid us in appreciating the peculiar difficulties and apparent contradictions met with in this subject.

Shortly after Galileo established the laws of falling bodies, further experiment showed that water and other inelastic fluids followed the same law. Proven that the velocity of water flowing through a properly shaped orifice was equal to the velocity which a body would acquire in falling a distance equal to the head, it followed that the velocity of efflux varied as the square roots of the heads. This law was applied to elastic fluids, such as air, as well, and crude experiments were made on a bladder first filled with air and then with water. Under the same pressure it was found that

it required 25 times as long to empty the bladder when filled with water as it did when filled with air; consequently the velocity of flow of air was considered to be 25 times that of water.

In a quaint old paper read by a certain Dr. Papin before the Royal Society in 1686 the following is stated as the general law deduced from theory and experiment. "Of different liquors flowing from an orifice bearing the same pressure, those lighter in specie must acquire the greater swiftness and their velocities are to one another as the roots of the specific gravities of the said liquors. As much as the Sq. Rt. of specific gravity of water doth exceed the square ? W.D.S. root of the specific gravity of air so much in proportion will the velocity of air exceed the velocity of water." This law was accepted for 115 years and it was not until late in the 19th century that further experiments were made.

In 1843 St. Venant and Wantzel made the first real advance and in their report & a give the first real intimation of one of the well known results of modern experiment. They make two important statements.

(1) When the pressure which acts in an opposite direction to the flow of the gas descends below three fifths of the pressure which causes the flow, the efflux instead of increasing diminishes.

(2) A gas cannot flow into a vacuum whatever be its pressure. These two conclusions so apparently opposed to all reason were too much for the experimenters and after a second series of experiments St. Venant says, "Our experiments have proved that those singular results have no more reality than they had of probability." As late as 1846 Prof. Graham of the Royal Society announced that his

experiments left no doubt as to the truth of the general law that different gases pass through minute apertures in times which are inversely as the square roots of their specific gravities. Experimenting and investigating along the same line in 1847, Mr. William Fronde, M. Inst. C. E., developed a theory of flow by which he proved that elastic fluids under pressure cannot flow into a perfect vacuum. His method of experiment was similar to that of St. Venant, Wantzel and Graham. From 1854 to 1862 Prof. Thomson and Dr. Joule made a series of experiments on the flow of air to determine the velocity through small orifices in thin plates. In Dr. Joule's report he gives a hint of one fact which has since proved of considerable importance. He says, " I have not been able to detect any effect due to vibration of the issuing stream. . . . there can be no doubt that vibration constituting sound will be able to travel back through the air rushing through the orifice if its velocity be not greater than 1090feet per second. I have failed, however, to discover any sensible influence from this cause on the velocity of efflux."

Robert D. Napier in 1866 was the first to state what we know now to be the actual conditions under which steam or any gas flows from a higher to a lower pressure. His important conclusion and the one which we know to be approximately true is, as he states it. "Steam at a pressure of two or more atmospheres will rush from a boiler through an orifice or short tube into the air exactly at the same rate as it will into a vacuum or into any pressure less than that of the atmosphere; and generally a gas of any given pressure will rush from one vessel into another containing a gas of half the pressure at the same rate as if there were a perfect vacuum in the recipient

vessel or any intermediate pressure between a vacuum and half the pressure in the cistern, both pressures being taken from zero." This conclusion, it is seen, had been once reached by St. Venant and Wantzel twenty years before and afterwards rejected by them as absolutely impossible. At this time Napier's experiments provoked heated discussions, but nevertheless the truth of his deductions was soon established as a scientific fact, Prof. Rankine being one of the first to refute arguments brought against it and one of the ardent supporters of the new theory.

The German scientists Zeuner, Fliegner and Grashof have since developed the theory and experimented more fully. In conclusion it may be said that at the present time the laws of the flow of gases through orifices in thin plates are known and the theory is supported by experiment.

Briefly stated the law of flow is as follows: When a gas flows through an orifice from a reservoir in which there exists a pressure P_1 into a vessel in which there exists a lower pressure P_2 the quantity of gas passing through the orifice continously increases as P_2 decreases until P_2 reaches the value .527 P_1 for air or .58 P_1 for steam. Any decrease of P_2 below this point has no effect on the quantity of gas flowing through the orifice in a given time. In other words when the maximum flow occurs $\frac{P_2}{P_1}$ = .527 for a gas or .58 for steam. As to the laws of flow in tubes of varying cross section and the phenomena taking place in the tube itself, much uncertainty exists and it is along this line that modern experimenters and scientists are working. The general law of flow through an orifice and through the ideal nozzle will be taken up in the next chapter.

CHAPTER 2.

The General Equation of Flow for Fluids.

The laws of flow for incompressible fluids are well understood and it will be of assistance to consider the general law of flow as applied to them, first deriving a general differential equation of motion which may be applied to any fluid, liquid or gas.

According to Bernoulli's law, in the case of frictionless flow the energy of the fluid is the same at all cross sections of the pipe or channel, differing only in kind, but not in total amount. This energy is of three kinds; (1) Mechanical Energy due to motion, (2) Pressure Energy due to pressure, (3) Intrinsic Energy due to molecular state. Each one of the three elements making up the total energy may be expressed in terms of pressure, volume, and velocity. Assuming unit mass to be flowing we have the three forms of energy expressed as follows:

$$\text{Mechanical Energy} = \frac{u^2}{2\,g}$$

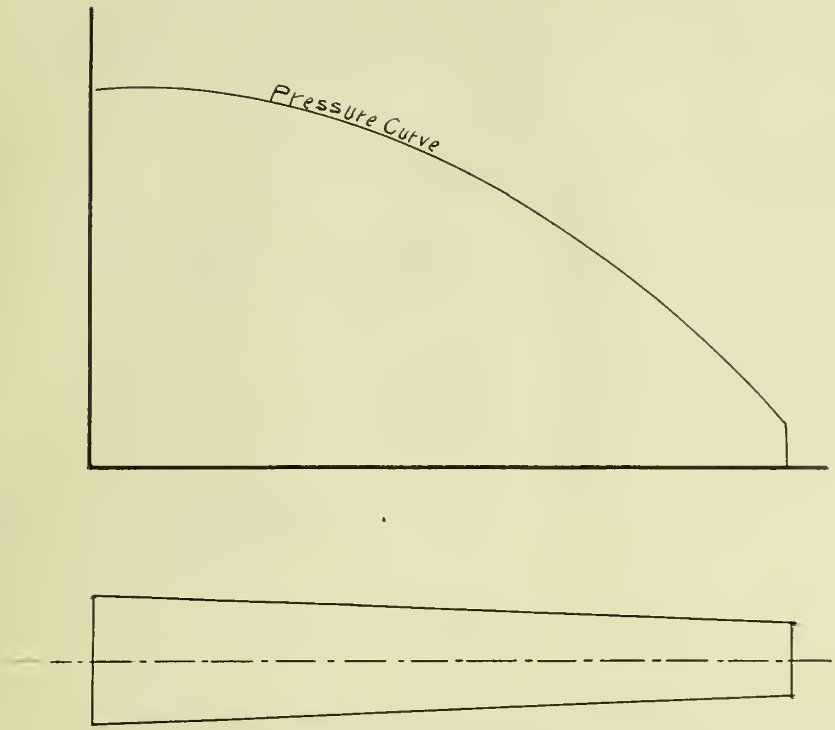
$$\text{Pressure Energy} = PV$$

$$\text{Internal Energy} = U$$

Our general differential equation is then, according to Bernoulli's law

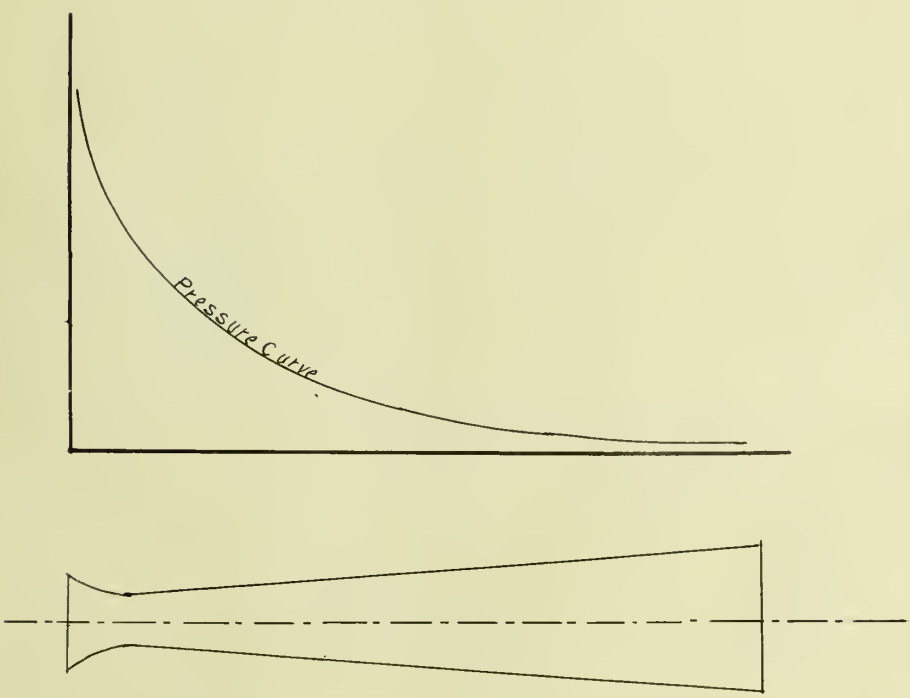
$$dE = \frac{u\,du}{g} + d(PV) + dU \quad \text{------------(1)}$$

Considering the fluid to be steadily flowing in the direction of the axis X we have as the equation of motion



Nozzle for Increasing Velocity of an Inelastic Fluid

FIG 1



Nozzle for Increasing Velocity of an Elastic Fluid

FIG 2

$$\frac{dE}{dx} = \frac{udu}{g\,dx} + \frac{d(PV)}{dx} + \frac{dU}{dx} \text{ --------- } (2)$$

But in the case of frictionless flow as assumed $\frac{dE}{dx} = 0$ and

$$\frac{UdU}{g\,dx} = -\frac{d(PV)}{dx} - \frac{dU}{dx} \text{ --------------- } (3)$$

Now for an inelastic fluid, such as water, the molecular state remains the same and the term $\frac{dU}{dx} = 0$ giving as a final equation for an inelastic fluid

$$\frac{UdU}{g\,dx} = -\frac{d(PV)}{dx}$$

in which V as we know is constant. Integrating

$$\frac{U^2}{2g} = -\int_{P_1}^{P_2} V\,dP = V(P_1 - P_2) = \frac{P_1 - P_2}{\delta}$$

in which δ represents the density of the liquid $= \frac{1}{V}$ Letting $\frac{P_1 - P_2}{\delta} = h$ we get the familiar equation of hydraulics $U^2 = 2gh$

We may from this equation deduce the form of nozzle necessary to increase the velocity of flow. If M is the mass flowing per second $= \frac{W}{g}$ then $\frac{W}{g} = U\delta F$ in which F is the area of the nozzle or $F = \frac{W}{gU\delta}$ g and δ are constants consequently F varies inversely as u and a nozzle for increasing the velocity should be one of gradually decreasing cross section as shown in the Fig. 1. Our differential equation shows that along such a nozzle P falls as u increases.

It was from this equation that the early investigators drew the apparently logical conclusion that, since U varied as $\sqrt{h}$ and h varied inversely as δ, that U varied inversely as $\sqrt{\delta}$, or expressed in words, the velocity of flow of fluids varies inversely as the square root of the density.

Examining the general equation $\frac{Udu}{g\,dx} = -\frac{d(PV)}{dx} - \frac{dU}{dx}$

closely, we see that this conclusion is unwarranted in the case of elastic fluids, because it cannot be integrated as simply as before for two reasons

(1) $\frac{dU}{dX}$ is not zero.

(2) V is not constant.

We shall see however that the resulting equation will take the same form

$$\frac{U^2}{2g} = -\int_{P_1}^{P_2} V\,dP$$

in which V is a variable dependent upon P.

To study the flow of an elastic fluid such as steam or gas we must return to the fundamental equations of thermodynamics in order to determine the significance of the term $\frac{dU}{dX}$ and the effect of V as a variable. Our fundamental equation is

$$dQ = dU + dW$$

or we may write it

$$dQ = dU + P\,dv$$

Applying it under the conditions under which the general equation of motion holds, viz.:

$$\frac{dE}{dX} = 0$$

or without the addition of energy from an external source we have

$$dQ = 0$$

and $$dU = -P\,dv$$

the familiar statement of adiabatic flow.

Going back to our original equation of motion

$$\frac{U\,dU}{g\,dx} = -\frac{d(PV)}{dx} - \frac{dU}{dx}$$

and substituting for dU its equal, $-Pdv$ we have

$$\frac{U\,dU}{g\,dx} = -\frac{P\,dV}{dx} - \frac{V\,dP}{dx} + \frac{P\,dV}{dx}$$

$$\frac{U\,dU}{g\,dx} = -\frac{V\,dP}{dx}$$

$$\frac{U^2}{2g} = -\int_{P_1}^{P_2} V\,dP$$

It may seem surprising at first that the equation takes precisely the same form when ready for integration although the term $\frac{dU}{dx}$ is not zero. The fact is easily explained. All the energy is being converted into mechanical energy of flow, both equations express this fact. The additional term $\frac{dU}{dx}$ is accounted for in the variable V which depends for its value on P and by P and V together the intrinsic energy is determined, being expressed as $\frac{d(PV)}{K-1}$

From the adiabatic relation

$$P_2 V_2^K = P_1 V_1^K$$

we may evaluate the integral $-\int_{P_1}^{P_2} V\,dP$ and find

$$\frac{U^2}{2g} = \frac{K}{K-1}\left(P_1V_1 - P_2V_2\right)$$

Eliminating V_2

$$\frac{U^2}{2g} = \frac{K}{K-1} P_1V_1\left\{1 - \left(\frac{P_2}{P_1}\right)^{\frac{K-1}{K}}\right\}$$

$$U = \sqrt{\frac{2gK}{K-1} P_1V_1\left\{1 - \left(\frac{P_2}{P_1}\right)^{\frac{K-1}{K}}\right\}}$$

If now we consider M to be the mass flowing per second from the nozzle of orifice and F the area of the orifice $M = \frac{FU}{V_2}$

$$M = \sqrt{\frac{2gKP_1}{(K-1)V_1}} \cdot \sqrt{\left(\frac{P_2}{P_1}\right)^{\frac{2}{K}} - \left(\frac{P_2}{P_1}\right)^{\frac{(K-1)}{K} + \frac{2}{K}}}$$

where is F? W.D.S

Hence along a steady stream since M is constant the above equation (4) gives a relation which must hold between F and P. Differentiating F with respect to P and writing $\frac{dF}{dP} = 0$ we get the relation

$$\frac{P_2}{P_1} = \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}}$$

and with $K = 1.408$ $\frac{P_2}{P_1} = .527$

It thus appears that as long as P falls the section continously diminishes until it reaches a minimum value when $P_2 = .527 P_1$ and then increases again. We note too that our equation of motion

$$\frac{UdU}{gdx} = -\frac{VdP}{dx}$$

shows that as long as P falls the velocity increases and the shape of the nozzle to increase the velocity should be that shown in Fig. 2. This nozzle may be called the ideal nozzle because in it we have supposed the widening of cross-section to take place at just the proper point, that is, where the fluid has expanded to the pressure .527 of the initial pressure. The effect of not widening the cross-section or widening it at some point other than the critical point and the effect of back pressure at the mouth of the nozzle will be considered later on. We have now to investigate the velocity in the particulat case when

$$2P_1^{\frac{K-1}{K}} = (K+1)P_2^{\frac{K-1}{K}}$$

Taking our general velocity equation

$$U = \sqrt{\frac{2gK}{K-1}P_1V_1\left\{1-\left(\frac{P_2}{P_1}\right)^{\frac{K-1}{K}}\right\}}$$

and substituting in it this particular value of P_2 we get

$$U = \sqrt{\frac{2gK}{K-1}P_1V_1\left\{1-\frac{2}{K+1}\right\}}$$

$$U = \sqrt{\frac{2gK}{K+1}P_1V_1}$$

Since

$$\frac{2}{K+1} = \left(\frac{P_2}{P_1}\right)^{\frac{K-1}{K}} = \frac{T_2}{T_1} = \frac{P_2V_2}{P_1V_1}$$

$$U = \sqrt{2gK\frac{T_2}{T_1}P_1V_1} = \sqrt{gKP_2V_2}$$

or stated in words, the velocity of the gas in the narrowest cross-section of the nozzle is the velocity which sound would have in the gas in the state at that particular point, provided conditions are such that P_2 may fall to $.527\ P_1$ Peabody Thermodynamics P. 62

Let us return for a moment to our equation expressing the quantity flowing

$$M = \frac{FU}{V_2}$$

$$M = \sqrt{\frac{2gKP_1}{(K-1)V_1}}\sqrt{\left(\frac{P_2}{P_1}\right)^{\frac{2}{K}} - \left(\frac{P_2}{P_1}\right)^{\frac{K+1}{K}}}$$

Suppose we consider P_2 to be indefinitely reduced, the quantity $\sqrt{\left(\frac{P_2}{P_1}\right) - \left(\frac{P_2}{P_1}\right)^{\frac{K+1}{K}}}$ approaches 0 and at the limit is equal to 0 and we find $M = 0$, in other words a gas cannot flow into a perfect vacuum. This we recognize as the conclusion reached by William Fronde and St. Ve-

nant and Wantzel in their early experiments. This conclusion is unreasonable and not supported by experiment. The reason is simple; P_2 cannot be indefinitely reduced, that is the pressure at the narrowest cross section of the nozzle has a minimum value below which it cannot fall, no matter how low the back pressure may be.

We may explain this fact as follows. Let us imagine the nozzle to be cut off at the narrowest section as shown in the figure. We found the pressure here was defined by the relation

$$(K+1)P_2^{\frac{K-1}{K}} = 2P_1^{\frac{K-1}{K}}$$

that is, the pressure in the narrowest cross section, provided it is free to fall along the nozzle is determined by the initial pressure and is independent of the pressure in the receiving vessel P_0 provided P_0 be less than the value of P_2 determined from P_1. If P_0 be greater than the critical value of P_2, the only possible condition neglecting impact, is that $P_0 = P_2$ and our general equation determining the mass M flowing will hold.

$$M = F\sqrt{\frac{2gKP_1}{(K-1)V_1}}\sqrt{\left(\frac{P_2}{P_1}\right)^{\frac{2}{K}} - \left(\frac{P_2}{P_1}\right)^{\frac{K+1}{K}}} \text{ - - - - - - - - - - - - } (4)$$

In this equation we may substitute , the pressure in the receiving vessel for P_2, and determine M. In general we may say equation (4) holds for all values of $P_0 \geqq .527 P_1 = P_2$. Beyond this point P_2 is independent of P_0 and remains constant at the value $.527 P_1$. Consequently the second member of equation (4) never becomes 0 and a gas will flow into a perfect vacuum with a velocity determined by the initial state P_1V_1, the velocity, U, in this case being given by the equation

$$U = \sqrt{\frac{2gK}{K+1} P_1 V_1} \quad or \quad U = \sqrt{K g P_2 V_2}$$

In regard to an increase of section at the critical point, we may say that it will increase the velocity provided the pressure is allowed to drop from P_2 to a lower value P_0 . The mass flowing remains constant and is defined by the equation

$$M = \frac{FU}{V_2} = \frac{F\sqrt{K g P_2 V_2}}{V_2}$$

or if defined by the initial state

$$M = F\sqrt{\frac{2gK}{K+1} P_1 V_1}$$

or if a represent the velocity of sound in the medium at the critical section

$$M = \frac{Fa}{V_2}$$

CHAPTER 3.

Effect of Friction on Velocity.

So far, we have been considering the flow of steam under pure adiabatic conditions which, of course, cannot be realized in practice. Three things, under actual conditions, affect the flow of steam; interchange of heat between nozzle walls and steam, friction on the walls, and back pressure in the nozzle. As the velocity of the steam is so great, the interchange of heat between the nozzle walls and steam may be neglected, but the other two factors exercise a marked influence on the flow and the first of these, friction, will be considered in the following chapter.

In discussing this question we shall use again the fundamental equations of thermodynamics, but in a somewhat modified form. As has been shown the equation of adiabatic flow is

$$-\frac{dU}{dx} = \frac{u\,du}{g\,dx}$$

or expressing U as a function of pressure and volume

$$-\frac{K}{K-1}\,\frac{d(PV)}{dx} = \frac{u\,du}{g\,dx}$$

It will be well here to get clearly in mind what this equation means. It simply says, the rate at which the internal energy of the fluid is decreasing is equal to the rate at which the mechanical energy is increasing. We recognize this at once as the statement of the adiabatic condition, "All work is done at the expense of the internal energy." Putting this equation in differential form, we have

$$\frac{K}{K-1} d(PV) + \frac{UdU}{g} = 0 \quad ------------------(1)$$

The second equation to be used is somewhat more difficult to derive and interpret. We have seen that the function $\frac{K}{K-1} d(PV)$ may be expressed in the simpler form VdP provided the proper relation exists between V and P. Let us assume that the adiabatic relation holds between V and P and write

$$-\frac{VdP}{dx} = \frac{U'dU'}{gdx} + \frac{dR}{dx}$$

in which dR = work of friction. This equation is true inasmuch as it is a somewhat more general statement of equation (1). It says, the rate at which the internal energy is decreasing is equal to the rate at which the mechanical energy is increasing plus the rate at which friction work is done. We note here that $\frac{U'dU'}{gdX}$ is a rate different from $\frac{UdU}{gdX}$ if we consider $-\frac{VdP}{dX}$ the exact equivalent of $\frac{K}{K-1}\frac{d(PV)}{dX}$. Writing this equation in differential form

$$-VdP = \frac{U'dU'}{g} + dR$$

and subtracting it from equation (1) we get

$$\frac{UdU}{g} - \frac{U'dU'}{g} = dR$$

This equation states, that if flow occurs without friction and then with friction, the difference of the mechanical energies at any two cross sections under the two conditions measures the work of friction.

For the sake of convenience in handling the equations and deriving results we may make the following assumptions, viz.: $\frac{U'dU'}{g} = \frac{UdU}{g}$ but the two terms VdP and $\frac{K}{K-1} d(PV)$ are not equivalent, in other words the expansion of the fluid does not follow the law PV^K = con-

stant. This would naturally be the case in an irreversible process such as we are considering. Our two equations may now be written

$$\frac{K}{K-1}\, d(PV) + \frac{U\,dU}{g} = 0$$

$$V\,dP + \frac{U\,dU}{g} + dR = 0$$

Assuming a coefficient of friction such as is used in hydraulic formulae dR may be expressed as $\frac{\ell U^2}{2g}dx$ or since ℓ is an arbitrary constant simply $\frac{\ell U^2}{g}dx$ giving us in final form two equations

$$\frac{K}{K-1}\, d(PV) + \frac{U\,dU}{g} = 0 \text{ ----------------- } (1)$$

$$V\,dP + \frac{U\,dU}{g} + \frac{\ell U^2}{g}dx = 0 \text{ ---------- } (2)$$

In these equations V and P cannot be considered as connected by the equation PV^K = constant unless $\ell = 0$ as is shown by the following:

Subtracting (2) from (1) we have

$$\frac{K}{K-1}\, d(PV) - V\,dP = \frac{\ell U^2}{g}dx$$

Expanding and combining terms

$$\frac{KP\,dV + V\,dP}{K-1} = \frac{\ell U^2}{g}\,dx$$

Taking the equation PV^K = constant and differentiating it we find

$$KP\,dV + V\,dP = 0$$

and consequently if this law holds $\frac{\ell U^2}{g}dx = 0$ or $\ell = 0$

We may now compare equation (2) with the general equation of flow derived in the first chapter.

$$\frac{U^2}{2g} = -\int_{P_2}^{P_1} V\,dP \text{ ---------- } 1(a)$$

$$\frac{U^2}{2g} = -\int_{P_2}^{P_1} V\,dP - R \text{ ------- } (2)$$

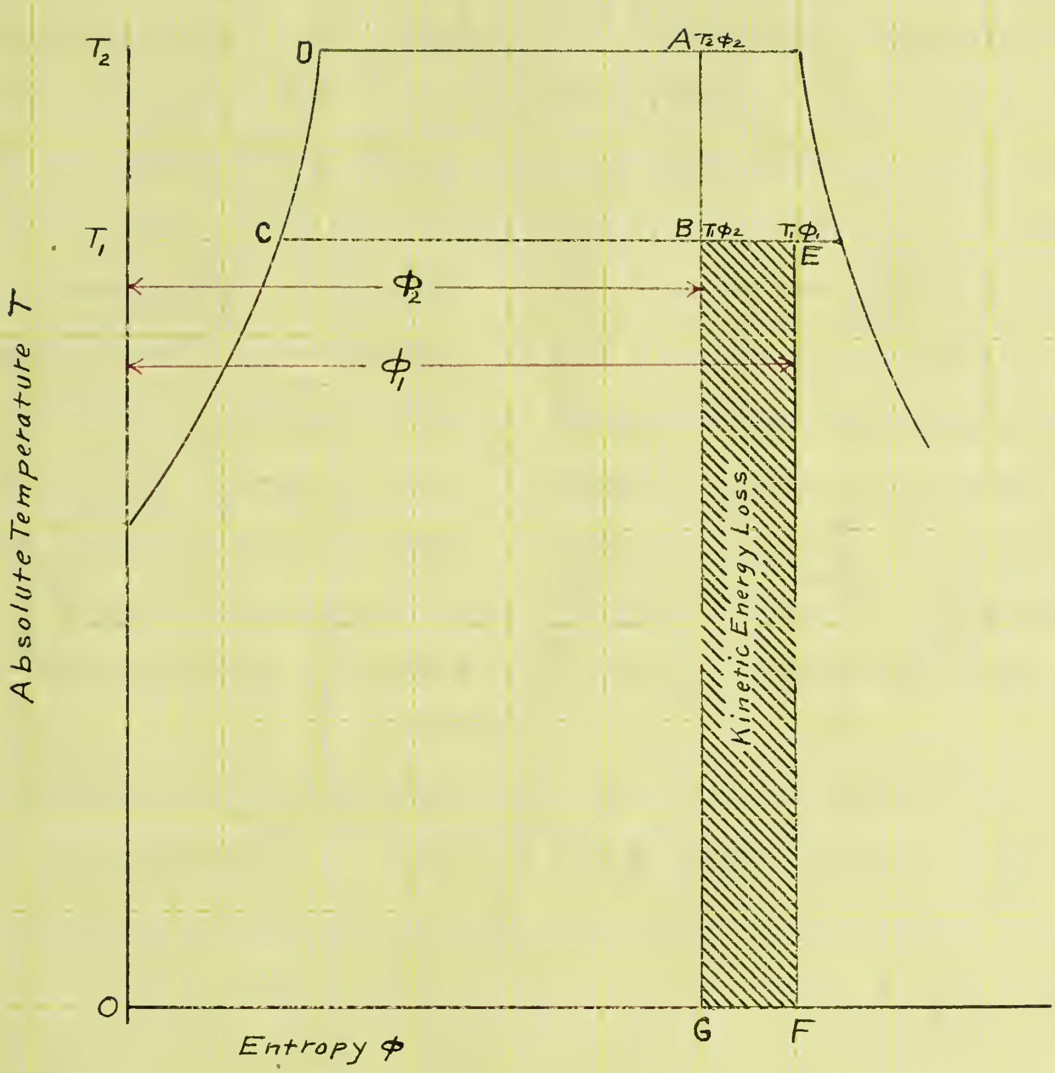


FIG. 3

$T\Phi$ Diagram Showing Friction Loss

Both these equations hold under the general adiabatic condition $dQ=0$ For equation 1 (a), the law PV^K = constant will hold; for equation (2), this law will hold only when $R=0$. In other words, there are two kinds of adiabatics, one without friction loss, and one with friction loss. In the adiabatic without friction, the increase of kinetic energy $\frac{U^2}{2g}$ is measured by the integral $-\int_{P_2}^{P_1} V dP$ evaluated under the law PV^K = constant. In the adiabatic with friction, the increase of kinetic energy is not as large inasmuch as a part of the available kinetic energy has been converted into heat and passes out of the end of the nozzle as heat. Consequently, $\frac{U^2}{2g}$ is measured not by $-\int_{P_2}^{P_1} V dP$ as before, but by $-\int_{P_2}^{P_1} V dP - R$ or $-\int_{P_2}^{P_1} V dP$ in which some law other than PV^K = constant connects the variables P and V.

The effect of friction work R and the value of R may be represented on the $T\phi$ plane as shown in Fig 3. P. 15(a). Suppose steam to exist at the nozzle entrance in the state $T_2\ \phi_2$ and consider it to expand under purely adiabatic conditions to the state $T_1\ \phi_2$ that is, down the vertical line AB. The work done or the increase of mechanical energy is the work value of the heat area $ABCD$ and in the PV coordinates would be represented by $\int_{P_2}^{P_1} V dP$. Now under our actual conditions we find that the steam does not reach the state $T_1\ \phi_2$ but is actually in the state $T_1\ \phi_1$ at the point E. All areas below the line CE represent lost heat or lost kinetic energy, consequently the loss of kinetic energy over and above that lost in adiabatic expansion is the area $BEFG$. Since the process is irreversible we know nothing of the path from the state $T_2\ \phi_2$ to $T_1\ \phi_1$.

So our only method of determining the increase of kinetic energy is to take the difference of total heats at the states $T_2\,\phi_2$ and $T_1\,\phi_1$ and find its mechanical equivalent. It should be noted here that the area $BEFG$ does not necessarily represent all the friction work done. There may be some of heat of friction converted into kinetic energy while the steam passes from A to E and this, of course, is not lost and so does not appear in the area representing loss of kinetic energy. Not knowing the path from A to E we are unable to draw the curve bounding this area.

Before leaving the question of friction we shall briefly consider its effect on the quantity of steam flowing through the nozzle, since the impact involves both a mass and a velocity factor. If we take our frist general differential equation and integrate it

$$\int_{P_2V_2}^{P_1V_1} \frac{K}{K-1}\, d(PV) = \frac{K}{K-1}\left(P_1V_1 - P_2V_2\right) = \frac{U^2}{2g}$$

or $$U^2 = \frac{2gKP_1V_1 - 2gKP_2V_2}{K-1}$$

In the first chapter we have shown that the product KgP_1V_1 is a_1^2, when a_1 represents the velocity of sound in the medium in the state P_1V_1 . Introducing this quantity a_1^2 we write

$$U_1^2 = \frac{2a_1^2 - 2a_2^2}{K-1}$$

If we consider P_2V_2 to represent any state along the nozzle, we may write PV in the place of P_2V_2 and a in the place of a_2 and find the general expression for the velocity at any point

$$U^2 = \frac{2a_1^2 - 2a^2}{K-1}$$

At the nozzle entrance $U = 0$ and $a_1 = a$ which is evidently true. As the steam passes on through the nozzle U increases and at some point reaches the value a. In this case

$$a^2 = \frac{2(a_1^2 - a^2)}{K-1}$$

$$a^2 = \frac{2a_1^2}{K+1}$$

Expressing this again in terms of $P_1 V_1$ and U we have

$$U = \sqrt{\frac{2gKP_1V_1}{K+1}}$$

From this equation we draw the following conclusion:

If the steam velocity, U, in the nozzle reaches the value $U = \sqrt{KgPV} = a$, it remains at that value in that cross section for any given initial state $P_1 V_1$. In general, in a steam nozzle in which the velocity is increasing, there is a certain cross section in which the velocity has a fixed value determined by the initial state. As a consequence of this, there is a certain maximum quantity of steam which can flow through a given nozzle from a fixed initial state $P_1 V_1$.

Examining the equation

$$U = \sqrt{\frac{2gKP_1V_1}{K+1}}$$

again, we see that it is the same one which we derived under frictionless adiabatic conditions, when the cross section was at its minimum value. We found, too, that at this section the pressure bore a fixed ratio to the initial pressure given by the equation

$$\frac{P_2}{P_1} = \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}}$$

We see now that the pure adiabatic is a special case of the friction adiabatic and we may re-state the former conclusion as follows:

In any nozzle there is a certain cross section in which the pressure and velocity are fixed by the initial state. If the expansion is purely adiabatic, this cross section will be the neck of the nozzle and the pressure there will be

$$P_2 = \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}} P_1$$

Under actual conditions these last two statements are not necessarily true. In either case, there is a certain maximum quantity of steam which can pass from an initial state $P_1 V_1$ through a nozzle of given cross section, this maximum quantity being less in the case of frictional resistance owing to a reduction of velocity.

The relation of the velocity of sound $\sqrt{KgPV}$ to the actual velocity of the steam is a complicated and peculiar one and will be discussed in the next chapter.

CHAPTER 4.

The Relation of the Velocity of Sound to the Velocity of Steam.

The function $\sqrt{KgPV}$ which has been used in the preceding chapters appears in a peculiar relation to the velocity of the steam U, as soon as we try to find the law of the variation of U. The function $\frac{du}{dx}$ we already found to be dependent upon $\frac{dP}{dx}$ so in studying the law of the variation of $\frac{dU}{dx}$ we will find first the general expression for $\frac{dP}{dx}$.

In addition to the equations already derived, we will use one more, the equation of steady flow, $GV = FU$ in which G = number of pounds of steam flowing in any cross section per second, V, the volume per pound, F, the area of cross section, and U, the velocity along the nozzle. Differentiating this equation, we have

$$\frac{dv}{v} = \frac{dU}{U} + \frac{dF}{F}$$

The equations which we now have are the following:

$$\frac{K}{K-1}d(PV) + \frac{UdU}{g} = 0 \qquad \text{--------------} \quad (1)$$

$$VdP + \frac{UdU}{g} + \frac{\zeta U^2}{g}dx = 0 \qquad \text{------------} \quad (2)$$

$$\frac{dV}{V} = \frac{dU}{U} + \frac{dF}{F} \qquad \text{-----------------} \quad (3)$$

$$a^2 = KgPV \qquad \text{------------------} \quad (4)$$

Eliminating dV and dU we get the following important equation

$$\frac{dP}{dX} = \left\{ -\zeta \frac{U^2}{a^2}\left(1 - \frac{KU^2}{U^2-a^2}\right) - \frac{U^2}{U^2-a^2}\ \frac{dF}{FdX} \right\} KP - - - - (5)$$

Letting $\alpha = \frac{U^2}{a^2}(K-1)+1$ we may write equation (5) in the much simpler form

? W.P.S.

$$\frac{dF}{dX} = \frac{\alpha\zeta - \frac{dF}{FdX}}{U^2-a^2}\ KPU^2$$

Since F varies as R^2 in a cylindrical tube

$$\frac{dF}{FdX} = \frac{2\,RdR}{R^2dX} = \frac{2\,dR}{RdX}$$

and our equation becomes

$$\frac{dP}{dX} = \frac{\alpha\zeta - \frac{2dR}{RdX}}{U^2-a^2}\ KPU^2 - - - - - - - - - - - (5)$$

Before going farther it will be well to get clearly in mind the different terms in the right hand member of equation (5). U is the actual velocity of the steam along the nozzle defined ~~be~~ by the relation found in chapter 1, $U^2 = \frac{2Kg}{K-1}(P_1V_1 - PV)$, K being the ratio $\frac{C_P}{C_V}$. a is the velocity which sound would have in the steam in the state PV and is defined by the equation $a^2 = KgPV$. If K is taken at the value 1.33 for steam then we find $\alpha = .33\frac{U^2}{a^2}+1$ (ζ as experimentally determined by Stodola for one of his nozzles was .039 which will give some idea of its general value).

The function which we are to investigate is $\frac{dU}{dX}$. The first question to be answered is, what are the conditions of maximum velocity? Taking equation (2) and placing $\frac{dU}{dX} = 0$ we find

$$-\frac{dP}{PdX} = \frac{K\zeta U^2}{a^2}$$

Applying the same conditions to equations (1) and (3), we find the following relations to hold

$$\frac{dP}{P} = -\frac{dV}{V} \quad \textit{from equation (1)}$$

$$\frac{dV}{V} = \frac{dF}{F} \quad \textit{from equation (3)}$$

Hence

$$\frac{dF}{F\,dX} = K\mathcal{L}\frac{U^2}{a^2}$$

From these equations we draw the following conclusions:

(a) When the maximum or minumum velocity is reached the pressure is falling.

(b) When the maximum velocity is reached the cross section is increasing.

(c) If $\mathcal{L} = 0$, then when $\frac{dU}{dX} = 0$, $\frac{dP}{dX} = 0$, that is if no friction work is done, the velocity reaches a maximum when the pressure reaches a minumum.

In general $\frac{dP}{dX}$ is not 0 when $\frac{dU}{dX} = 0$ and hence the pressure is not at its minimum when the velocity is a maximum, but is falling toward it and the pressure curve is gradually becoming parallel with the axis of X.

If we take the case of a cylindrical straight nozzle, we find some interesting relations. In this case $\frac{dF}{F\,dX} = 0$ and equation (5) becomes simply $\frac{dP}{dX} = \frac{\alpha\mathcal{L}}{U^2 - a^2} KPU^2$. We see at once that whatever the values of α and $\mathcal{L}$ ~~are~~, the pressure will rise for $U > a$ and fall for $U < a$. Consequently in a straight tube we would expect the pressure to fall steadily until the value a was reached. This is exactly what Zeuner found to be true in his experiments on

the flow of air through straight tubes. If we examine the conditions of maximum velocity here we shall find some important relations. Our three conditions are

$$\frac{dU}{dx} = 0$$

$$-\frac{dP}{Pdx} = K\zeta\frac{U^2}{a^2}$$

$$\frac{dF}{Fdx} = K\zeta\frac{U^2}{a^2}$$

Now we know that $\frac{dF}{Fdx} = 0$ hence at this point $\frac{dP}{Pdx} = 0$ and these two equations can be true only when $\zeta = 0$ or <u>at the mouth of the nozzle</u>. Consequently in a straight cylindrical tube the maximum velocity and minimum pressure are reached at the same point and that point is the mouth of the nozzle. This also agrees with the statement that the maximum velocity is reached when the cross section is increasing and shows that in a divergent tube the point of maximum velocity may or may not be the mouth of the nozzle.

There is another question which we may investigate here, viz.: the maximum possible velocity in a straight tube. We have already found that in a straight tube the minimum pressure and maximum velocity take place at the same point. We will now try to find what this maximum velocity is. To do this we will take instead of $\frac{dP}{dX}$ the function $\frac{dU}{dX}$ and get an expression for $\frac{dU}{dX}$ in terms of P, V, U, F and X. Taking equation (1) and integrating it, we have

$$\frac{K}{K-1}P_1V_1 - \frac{K}{K-1}PV - \frac{U^2}{2g} = 0$$

From the equation $GV = FU$ we get $V = \frac{FU}{G}$ and substituting

this value of V we find

$$\frac{PF}{G} = \frac{P_1 V_1}{U} - \frac{K-1}{2gK}\,U \qquad (7) \qquad \text{W. D. S.}$$

From equation (2) by making the same substitution we get

$$\frac{FdP}{G} + \frac{dU}{g} + \frac{\ell U}{g}\,dX = 0$$

From equation (7) by differentiation

$$\frac{FdP}{g} + \frac{PdF}{G} = -\left(\frac{P_1 V_1}{U^2} + \frac{K-1}{2gK}\right)dU$$

Subtracting equation (8) from this we get

$$\frac{PF}{G}\,\frac{dF}{F} - \frac{dU}{g} = -\left(\frac{P_1 V_1}{U^2} + \frac{K-1}{2gK}\right)dU + \ell\frac{U}{g}dX$$

Substituting in this the value of $\frac{PF}{G}$ from equation (7)

$$\frac{dU}{UdX} = \frac{\left(\frac{P_1 V_1}{U^2} - \frac{K-1}{2gK}\right)\frac{dF}{FdX} - \frac{\ell}{g}}{\frac{K+1}{2gK} - \frac{P_1 V_1}{U^2}} \quad \text{-------------} (9)$$

Now assuming that we have a straight tube, $\frac{dF}{FdX} = 0$ and we get the very simple equation

$$\frac{dU}{dX} = -\frac{\ell U}{\frac{K+1}{2gK} - \frac{P_1 V_1}{U^2}}$$

Examining this equation we find the same result derived as before namely, when $\frac{dU}{dX} = 0$, $\ell = 0$, that is, the velocity is a maximum at the mouth of the nozzle. Evidently, the denominator of this fraction may be equal to, greater than, or less than zero under the condition $\frac{dU}{dX} = 0$. We will examine these conditions separately. We have shown in the preceding discussion that when the maximum quantity of steam was flowing the velocity U is given by the equation

$$U = \sqrt{\frac{2gK}{K+1}P_1 V_1} \quad \text{or} \quad U = a$$

Examining our denominator above, we see that when $U = \sqrt{\frac{2gK}{K+1}P_1 V_1}$

our denominator is equal to zero, or $\frac{dU}{dX} = \frac{0}{0}$, that is, when the velocity is a maximum and the mass flowing a maximum the velocity at the end of the nozzle is the velocity of sound. Now let us suppose

$$\frac{K+1}{2gK} - \frac{P_1V_1}{U^2} > 0 \qquad \frac{P_1V_1}{U^2} < \frac{K+1}{2gK}$$

and $U > \sqrt{\frac{2gKP_1V_1}{K+1}}$ or $U > a$

Now this condition cannot be true, for when $U = a$ the maximum quantity is flowing, consequently if $U > a$ held true, a quantity larger than the maximum would have to flow which is impossible. The third condition is $\frac{K+1}{2gK} - \frac{P_1V_1}{U^2} < 0$ or $U < \sqrt{\frac{2gKP_1V_1}{K+1}}$ or $U < a$

This is perfectly possible and simply means that the back pressure is such that U never reaches its maximum value a and consequently the maximum quantity of steam is not flowing.

Before leaving the subject of a straight tube we may say that when the straight tube is placed at the end of a divergent tube, the velocity of the entering steam may be greater than the value $U = a$. Let us see the effect on the function $\frac{dU}{dx}$ for $U > a$. This means that our denominator is positive, hence $\frac{dU}{dx}$ is a decreasing function or U is a decreasing function of X , that is, when $\frac{dU}{dX} = 0$ we have a <u>minimum</u> and not a maximum at the end of the nozzle. We have already shown that the function $\frac{dU}{dX}$ is dependent on $-\frac{dP}{dX}$, consequently when U is decreasing P is increasing, and we have a confirmation of the statement of Stodola, viz.: "In straight tubes, the pressure, not considering back pressure, will rise or fall according as the steam velocity is greater or less than the velocity of sound."

Thus far, we have investigated the velocity in a straight nozzle, that is, in a tube in which $\frac{dF}{Fdx} = 0$. We shall now try to ascertain some of the laws in accordance with which flow will take place in a diverging tube. We will first go back to our three conditions of maximum velocity

$$\frac{du}{dx} = 0$$

$$\frac{dP}{Pdx} = -K\zeta\frac{U^2}{a^2}$$

$$\frac{dF}{Fdx} = K\zeta\frac{U^2}{a^2}.$$

To begin with $\frac{dF}{Fdx} = 0$ in two places, viz.: in the narrowest section of the nozzle and at the mouth of the nozzle. At the narrowest section ζ is not 0, and therefore $\frac{dF}{Fdx}$ having the value 0 is not equal to $K\zeta\frac{U^2}{a^2}$ there, our three conditions are not fulfilled, and $\frac{dU}{dx}$ is not zero; in other words, a maximum or a minimum velocity is not reached in the narrowest cross section. $\frac{dF}{Fdx}$ may be 0 also at the end of the nozzle and our three equations may hold, consequently we may have a maximum or a minumum velocity at the end of the nozzle. Here should be noted carefully the difference between the straight tube and the diverging nozzle. In the straight tube U cannot become equal to a except under the condition $\zeta = 0$. In the diverging tube since $\frac{dF}{Fdx}$ is not always 0, U may be equal to a under the condition $\frac{dF}{Fdx} = K\zeta$ provided $\frac{dU}{dx}$ is 0 at the same time. It would appear from this that when the condition $U = a$ was fulfilled within the tube that this was the maximum value of U. This is not necessarily true.

Taking the equation

$$\frac{dU}{U\,dx} = \frac{\left(\frac{P_1V_1}{U^2} - \frac{K-1}{2gK}\right)\frac{dF}{F\,dx} - \frac{b}{g}}{\frac{K+1}{2gK} - \frac{P_1V_1}{U^2}}$$

and imposing the two conditions $\frac{dF}{F\,dx} = Kb$ and $U = a$ we find $\frac{dU}{dx} = \frac{0}{0}$, an indeterminate. That is $\frac{dU}{dx} = 0$ is only one possible value satisfying the conditions $\frac{dF}{F\,dx} = Kb$ and $U = a$. Evidently, we shall have to determine the value of the function when it takes the form $\frac{0}{0}$ by differentiation. To write the differentiations more simply, we shall now write $\frac{d(\mathrm{Log}\,U)}{dx}$ instead of $\frac{dU}{U\,dx}$ and $\frac{d(\mathrm{Log}\,F)}{dx}$ instead of $\frac{dF}{F\,dx}$. Differentiating numerator and denominator of equation we get

$$\left\{\frac{d(\mathrm{Log}\,U)}{dx}\right\}^2 = -\left(\frac{d(\mathrm{Log}\,F)}{dx}\right)\left(\frac{d(\mathrm{Log}\,U)}{dx}\right) + \frac{2}{K+1}\,\frac{d}{dx}\left(\frac{d\,\mathrm{Log}\,F}{dx} - Kb\right)$$

Since when $U = a$, $\frac{d(\mathrm{Log}\,F)}{dx} = Kb$ the last term of this equation vanishes and the expression becomes

$$\left\{\frac{d(\mathrm{Log}\,U)}{dx}\right\}^2 = -\left(\frac{d(\mathrm{Log}\,F)}{dx}\right)\left(\frac{d(\mathrm{Log}\,U)}{dx}\right)$$

The solution of this equation gives for $\frac{d(\mathrm{Log}\,U)}{dx}$ two values; $\frac{d(\mathrm{Log}\,U)}{dx} = 0$ and $\frac{d(\mathrm{Log}\,U)}{dx} = -Kb$. These two equations give us a clear idea of the phenomena in the divergent tube. $U = a$ may be the maximum velocity reached as we have seen before with a <u>falling</u> pressure. The only other condition under which $U = a$ is $\frac{d(\mathrm{Log}\,U)}{dx} = -Kb$ and U is a <u>decreasing</u> function of X, that is, U has once passed the value a and now decreases to that value with a rising pressure, a circumstance which Stodola's pressure curves clearly show. Here again the difference between the straight tube and diverging nozzle

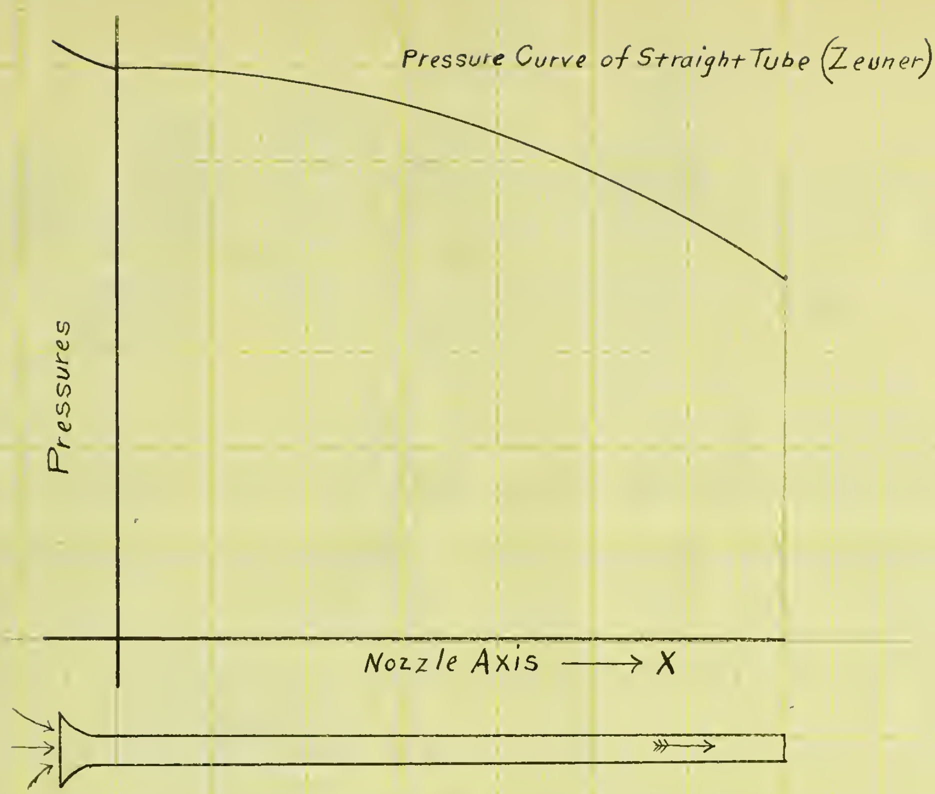

Pressure Curve of Straight Tube (Zeuner)
Pressures
Nozzle Axis ⟶ X

FIG 1

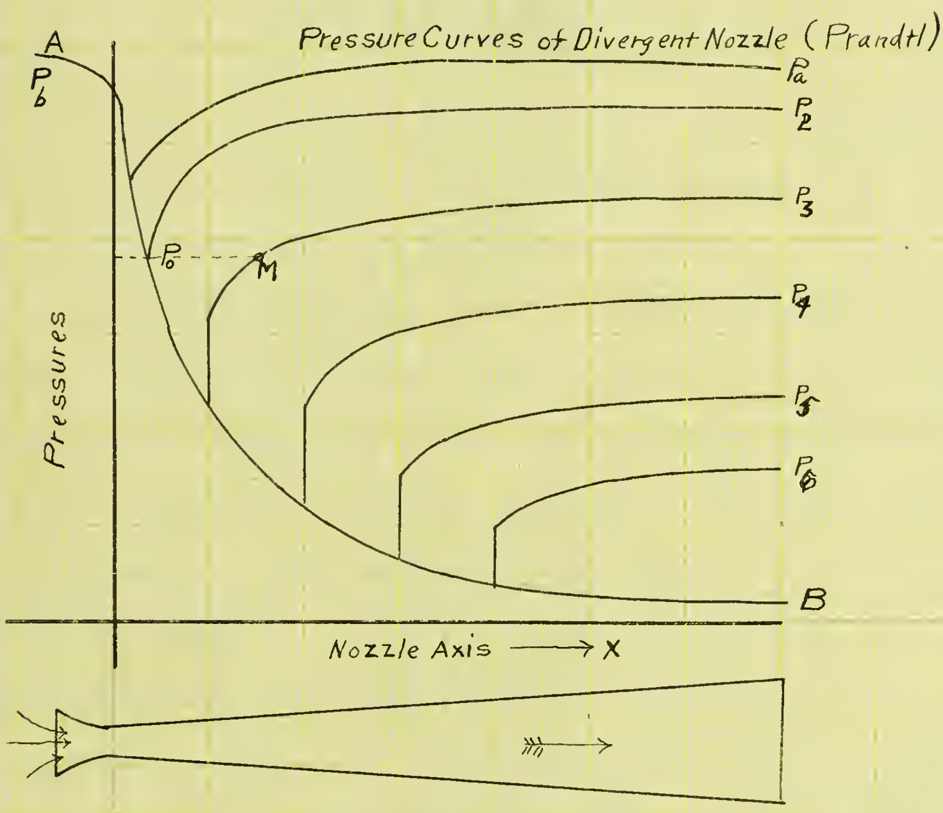
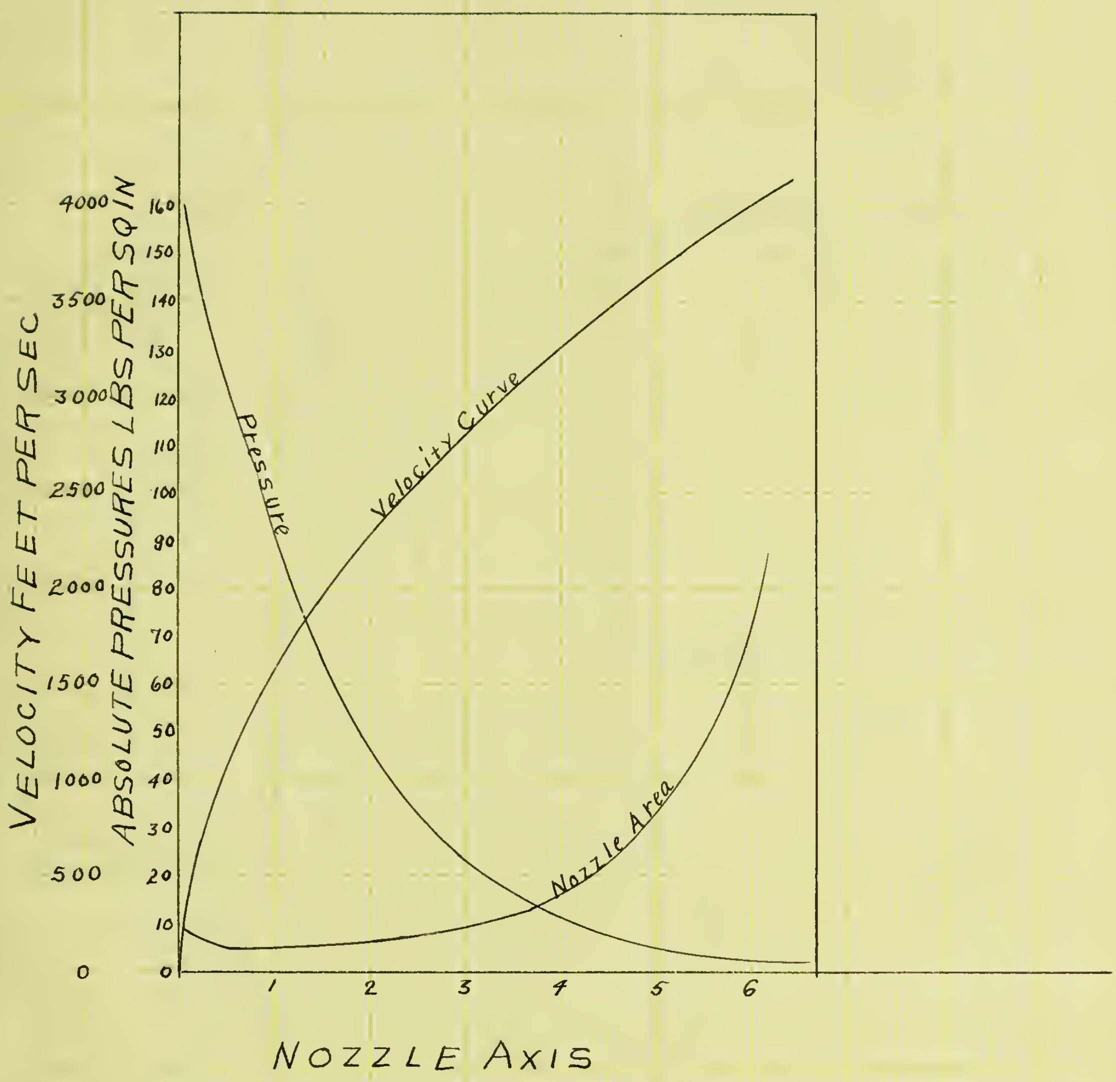

Pressure Curves of Divergent Nozzle (Prandtl)
A
P_b
P_a
P_2
P_3
P_o
M
P_4
P_5
P_6
B
Pressures
Nozzle Axis ⟶ X

FIG 2

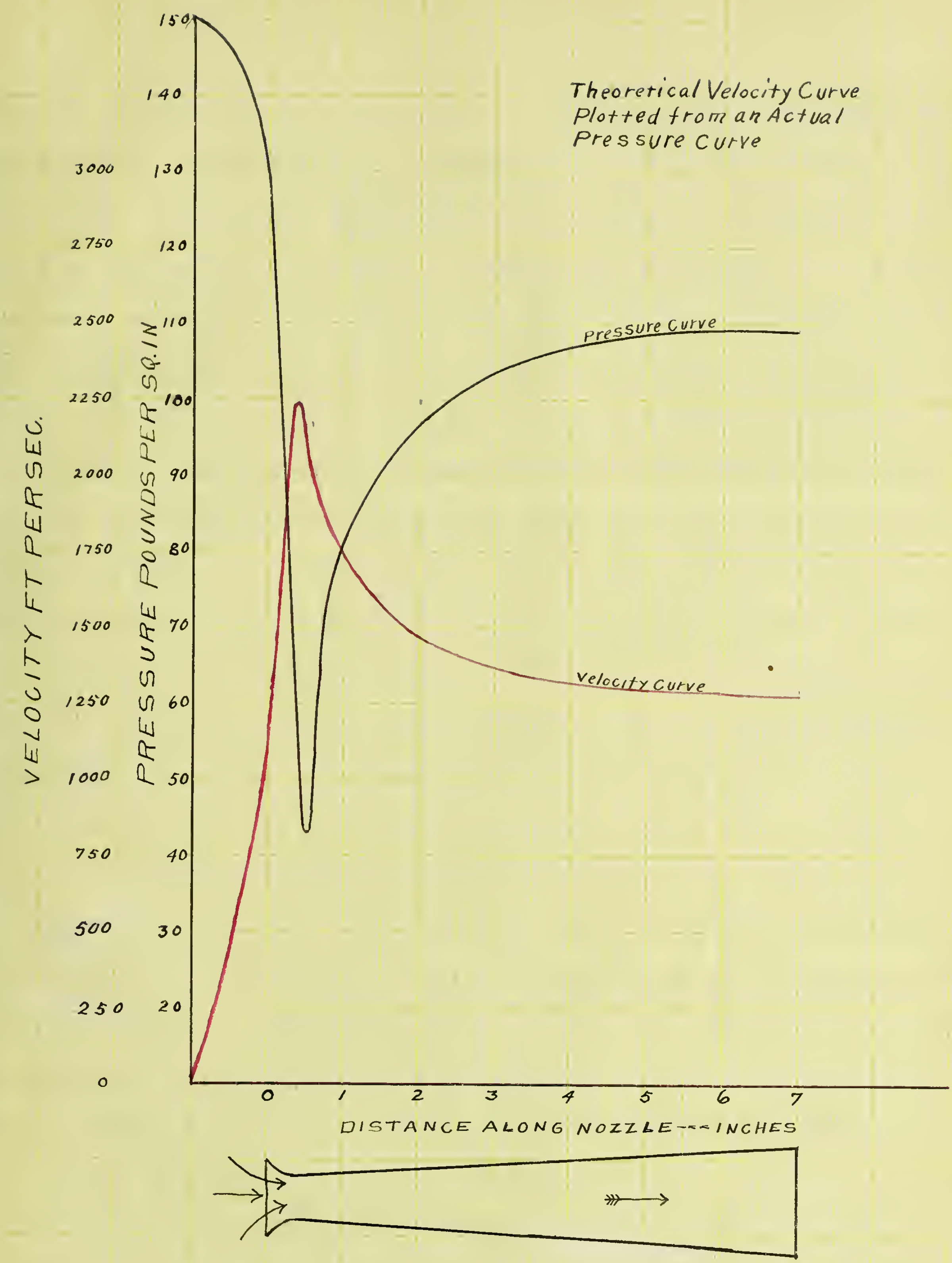
shows plainly. In the straight tube $\frac{dF}{Fdx} = 0$ throughout the tube and at the mouth $\ell = 0$ and the only possible solution for the above equation is $\frac{dU}{dx} = 0$, that is, a maximum value of U is reached when $U = a$.

A graphical representation of the foregoing equations by means of actual pressure curves will aid us in understanding the actual state of affairs in the nozzle. Fig. 1 shows the pressure curve for a straight tube as determined by Zeuner. The curve shows that the pressure falls continuously along the nozzle axis to the value of the back pressure, provided this back pressure is not less than the critical value (theoretically .58 P_1) and practically determined by the value of K from the equation $\frac{P_0}{P_2} = \left(\frac{2}{K+1}\right)^{\frac{K}{K-1}}$. The pressure at the mouth of the nozzle never falls below this value and the velocity never goes beyond the value $U = a$, no matter how far the back pressure may be reduced. If the back pressure P_a is above the critical value, the value $U = a$ is never reached. Fig. 2 shows the pressure curves for a diverging tube. Steam is flowing from the initial pressure P_1 to the various back pressures P_2, P_3, P_4, P_5, P_6, etc. The pressure falls along some curve AB similar to an adiabatic determined experimentally. At some point on this curve there is the pressure P_0 which is $.58\,P_1$ and here the steam velocity is $U = a$ $= \sqrt{\frac{2gKP_1V_1}{K+1}}$ or $\sqrt{KgPV}$. Now let us suppose that the back pressure P_a is so high that P never reaches the value P_0. Then U never reaches the value a. As shown by our equations, $\frac{dU}{dX}$ and $-\frac{dP}{dX}$ do not become zero at the same time but $\frac{dU}{dX} = 0$ while P is falling, consequently our velocity curve would reach a maximum just before the



THEORETICAL DESIGN OF NOZZLE
FOR
INCREASING VELOCITY OF STEAM

FIG 4.



Theoretical Velocity Curve Plotted from an Actual Pressure Curve
Pressure Curve
Velocity Curve
VELOCITY FT PER SEC.
PRESSURE POUNDS PER SQ. IN
3000
2750
2500
2250
2000
1750
1500
1250
1000
750
500
250
0
150
140
130
120
110
100
90
80
70
60
50
40
30
20
0
1
2
3
4
5
6
7
DISTANCE ALONG NOZZLE --- INCHES

FIG 3

pressure reached its minimum value. Next suppose a pressure P_2 is sufficiently low to allow the pressure to fall to the critical value P_0 At this point $U = a$, $\frac{d(\log U)}{dx}$ takes the form $\frac{0}{0}$ and is found to be either 0 or $-K\mathcal{L}$, that is, U may or may not be a maximum. If P drops to a value below the critical value, we can see plainly the point where $\frac{d(\log U)}{dx} = -K\mathcal{L}$ It is the point M on the pressure curve, for here, $P = P_0$, the critical value, and $U = a$ with a rising pressure.

Our equation gives no information as to the variation of U along the curve AB except to show that when the pressure falls, the velocity increases. As to the point where the pressure will begin to rise or whether it will rise at all, our equations give no information. Our conclusion is that this pressure curve and its attendant velocity curve are functions of the back pressure, and an equation must be found which will take this factor into account before the curve, representing the pressure and velocity can be found and plotted.

On the preceding page is shown a pressure curve with a corresponding velocity curve. The pressure curve is plotted from actual values determined by Stodola, and the velocities are calculated from the pressure drop. The curve shows clearly the relation of pressure and velocity and affords a good illustration of maximum velocity reached within the nozzle. The theoretically designed nozzle is shown in Fig. 4 P. 28(a) Whether this would cause the expansion line to approach more nearly to the adiabatic, we have as yet no means of knowing, nor do we know the effect of increased back pressure on such a nozzle, and herein lies a field for both theoretical and experimental investigation.

The above theory is by no means satisfactory for several reasons among which are the arbitrary assumptions, that K is constant, that the steam follows the walls of the nozzle as it passes out, and that we may discuss the phenomena in the nozzle with equations which do not involve the final pressure. However, the theory does indicate general laws which may be more fully and accurately stated after careful experiment. It also indicates along what lines experiment should be made to determine how closely the theory corresponds to actual conditions. Following are some conclusions which at present seem justified, and suggestions as to further experiment and investigation.

(1) Friction in the nozzle causes a reduction in the kinetic energy theoretically obtainable by a given expansion. Consequently, the theoretic velocity curve in Fig. 4 never corresponds to actual conditions.

(2) A maximum quantity of steam will flow through a straight tube from a fixed initial state, provided the final pressure be approximately the theoretic critical value .58 P_1 .

(3) A maximum quantity will flow through a diverging nozzle from a fixed initial pressure, provided the pressure in <u>some section</u> of the nozzle reaches the critical value. The value of the back pressure to cause this, the theory does not show.

(4) The maximum velocity is reached, in general, with an increasing cross section, that is, at the end of a straight tube and at the end of or within a diverging nozzle.

(5) Maximum velocity is reached with a falling pressure. Maxi-

mum velocity and minimum pressure are reached at the same time only, when friction work is 0.

The following are some practical suggestions for further investigation.

(1) The determination of actual pressure curves for nozzles of varying degrees of divergence in order to ascertain, if possible, the proper Value of the function $\frac{d(\log F)}{dx}$ or $\frac{dF}{dx}$.

(2) An accurate determination of the velocity of the issuing jet and the the pressure at the mouth of the nozzle.

(3) The determination of the quality of the steam at the end of the nozzle so as to determine the actual kinetic energy loss.

(4) The experimental verification of conclusion (5) above.

(5) The determination of temperature curves in nozzles of varying divergence.

The theory, too, may well be further investigated and the following lines are suggested.

(1) Can a value of $\frac{dF}{dx}$ be found which will give the best results for a given expansion P_1 to P_2 ?

(2) Does the theory actually show that $\frac{du}{dx}=0$ at the mouth of a straight tube or is $\frac{du}{dx}$ indeterminate there?

(3) Can the final pressure in a diverging nozzle be theoretically found?

(4) Can we derive an equation taking into consideration the initial state P_1V_1 and the final state P_2V_2 ?

(5) Plot theoretic pressure curves.

(6) Investigate the laws of steam impact within the nozzle to

account for pressure rise.

(7) Investigate isentropic lines.

(8) Investigate the whole question with the ordinary steam symbols, h, p, E and x.

(9) Determine the actual total value of the heat of friction including that converted into mechanical energy.